(6K) Se $f \in L^p(X)$, $\mu(X) < \infty$, então $f \in L^r(X)$, $1 \leq r \leq p$.

Dica: $|f|^r \leq 1 + |f|^p$

Também, $\|f\|_r \leq \|f\|_p \, \mu(X)^s$, onde $s = 1/r - 1/p$

Res:

De fato, note que $|f|^r \leq 1 + |f|^p$

pois, se $|f(x)| \leq 1 \Rightarrow$

$$|f(x)|^r \leq 1 \leq 1 + |f(x)|^p$$

e se $|f(x)| \geq 1$, $|f(x)|^r \leq |f(x)|^p$, logo

$$|f(x)|^r \leq 1 + |f(x)|^p$$

Segue que

$$\int |f|^r \leq \int 1 + |f|^p = \mu(X) + \|f\|_p < \infty$$

Logo $f \in L^r$.

Além disso, como $|f|^r \in L^{p/r}$, temos

$$\frac{1}{p/r} + \frac{1}{q} = 1 \iff \frac{1}{q} = 1 - \frac{r}{p} = \frac{p-r}{p} \iff q = \frac{p}{p-r}$$

$$\int |f|^r = \int |f|^r \cdot 1 \leq \||f|^r\|_{p/r} \, \|1\|_{\frac{p}{p-r}}$$

$$= \left(\int (|f|^r)^{p/r}\right)^{r/p} \mu(X)^{\frac{p-r}{p}}$$

$$= (\|f\|_p)^r \, \mu(X)^{\frac{p-r}{p}}$$

$$\Rightarrow \|f\|_r \leq \|f\|_p \left(\mu(X)^{\frac{p-r}{p}}\right)^{1/r}$$

$$= \|f\|_p \, \mu(X)^{\frac{p-r}{pr}} = \|f\|_p \, \mu(X)^{\frac{1}{r} - \frac{1}{p}} \qquad \square$$

6.2. Prove que $l^p \subset l^s$, com $1 \leq \overset{2}{\overset{=}{p}} < \overset{3}{\overset{=}{s}} < \infty$ e
$\|x\|_s \leq \|x\|_p \ , \forall x \in l^s$

RES)

Seja $x = (x_n)_n \in l^p$.

Então $\exists N, \ \forall n \geq N, \ |x_n| \leq 1$

$$\Rightarrow |x_n|^s \leq |x_n|^p \ , \forall n \geq N$$

Logo

$$\sum_{k=1}^{\infty} |x_k|^s = \sum_{k=1}^{N} |x_k|^s + \sum_{k=N+1}^{\infty} |x_k|^s \leq \underbrace{\sum_{k=1}^{N} |x_k|^s + \sum_{k=N+1}^{\infty} |x_k|^p}_{< +\infty}$$

Logo $x \in l^s$.

Agora

AF 1: $\|x\|_p \leq 1 \Rightarrow \|x\|_s \leq 1$

De facto, $\|x\|_p \leq 1 \Rightarrow |x_n| \leq 1, \forall n$, logo

$$\|x\|_s^s = \sum_{n=1}^{\infty} |x_n|^s \leq \sum_{n=1}^{\infty} |x_n|^p = \|x\|_p^p \leq 1$$

AF 2: $\|x\|_s \leq \|x\|_p$

Pelo (1),

$$\left\| \frac{x}{\|x\|_p} \right\|_p = 1 \overset{PT.1}{\Rightarrow} \left\| \frac{x}{\|x\|_p} \right\|_s \leq 1$$

$$\Rightarrow \|x\|_s \leq \|x\|_p$$

6N) Seja $f \in L^{p_1}$, $f \in L^{p_2}$, $1 \leq p_1 < p_2 < \infty$
Prove que $f \in L^p$, $\forall p_1 \leq p \leq p_2$

RES)
Como $p_1 \leq p \leq p_2$, $\exists \theta$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$
Assim

$$1 = \frac{p\theta}{p_1} + \frac{p(1-\theta)}{p_2}$$

Logo

$$\|f\|_p^p = \int |f|^p = \int |f|^{\theta p} |f|^{(1-\theta)p}$$

Mas $|f|^{\theta p} \in L^{\frac{p_1}{\theta p}}$, $\left\| |f|^{\theta p} \right\|_{\frac{p_1}{\theta p}} = \|f\|_{p_1}^{\theta p}$

e

$$|f|^{(1-\theta)p} \in L^{\frac{p_2}{(1-\theta)p}}, \quad \| |f|^{(1-\theta)p} \|_{\frac{p_2}{(1-\theta)p}} = \|f\|_{p_2}^{(1-\theta)p}$$

Assim, como $\frac{1}{\frac{p_1}{\theta p}} + \frac{1}{\frac{p_2}{(1-\theta)p}} = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = 1$, por Hölder:

$$\|f\|_p^p \leq \| |f|^{\theta p} \|_{\frac{p_1}{\theta p}} \| |f|^{(1-\theta)p} \|_{\frac{p_2}{(1-\theta)p}} = \|f\|_{p_1}^{\theta p} \|f\|_{p_2}^{(1-\theta)p}$$

$$\Rightarrow \|f\|_p \leq \|f\|_{p_1}^{\theta} \|f\|_{p_2}^{(1-\theta)}$$

$\square$

6.D Seja $\varepsilon > 0$, $f \in L^p$. Então $\exists\, \varphi$ função simples tal que $\|f - \varphi\|_p < \varepsilon$

De fato, escreva $f = f^+ + f^-$.
Então, existe $(\varphi_n^+)_n$ sequência de funções simples tal que $\varphi_n^+ \to f^+$, $0 \leq \varphi_n^+ \leq f^+$, $\forall n$.
Assim, $|\varphi_n^+ - f^+|^p \to 0$ q.t.p.
e

$$|\varphi_n^+ - f^+|^p \leq 2^p (f^+)^p \leq 2|f|^p \in L^1$$

logo,

$$\lim_{n\to\infty} \int |\varphi_n^+ - f^+|^p = \int \lim_{n\to\infty} |\varphi_n^+ - f^+|^p = 0$$

i.e: $\exists\, N$ tq $\forall n \geq N$, $\int |\varphi_n^+ - f^+|^p < \varepsilon$
Análogo, p/ N', φ_n^- e f^-

Assim, pondo $n \geq \max\{N, N^*\}$, $\varphi_n = \varphi_n^+ - \varphi^-$ é simples e

$$\int |f - \varphi_n|^p = \int |f^+ - \varphi_n^+ - f^- + \varphi_n^-|^p$$

$$\leq 2^p \left(\int |f^+ - \varphi_n^+|^p + \int |f^- - \varphi_n^-|^p \right)$$

$$\leq 2^p (\varepsilon + \varepsilon) = 2^{p+1} \varepsilon$$

Como ε foi arbitrário, segue.

Note que vale para $p = \infty$: seja $M = \|f\|_\infty$

Então, pondo

$$E_k = \{x \in X ; \ k \cdot \varepsilon \leq f(x) < (k+1) \cdot \varepsilon\}$$

temos que existem finitos $I \subset \mathbb{Z}$ tais que $\bigcup_{k \in I} E_k = X$

[Basta tomar $I = [-N, N]$, onde $N > \frac{M}{\varepsilon} \in \mathbb{N}$

Daí, ponha

$$\phi(x) = \{ k \cdot \varepsilon \ , \ x \in E_k$$

Daí

$$|f(x) - \phi(x)| \leq \varepsilon \ , \ \forall x$$

pois

$$f(x) - \phi(x) \leq (k+1)\varepsilon - k \cdot \varepsilon = \varepsilon$$

$\leq$

$k \cdot \varepsilon - k \cdot \varepsilon$ $\quad \Rightarrow \|f - \phi\|_\infty \leq \varepsilon$

6.E) Se $f \in L^p$, $1 \leq p < \infty$, então $E = \{x \in X ; |f(x)| \neq 0\}$ é σ-finito.

De fato, note que $E = \bigcup_{n=1}^{\infty} E_n$ onde

$$E_n = \{x \in X \mid |f(x)|^p > 1/n\}$$

Logo, basta mostrar que $\mu(E_n) < \infty$, $\forall n$.
De fato, suponha, por absurdo, que $\mu(E_{n_0}) = +\infty$
Então

$$+\infty > \int |f(x)|^p \geq \int_{E_{n_0}} |f(x)|^p > \int_{E_{n_0}} 1/n = \frac{1}{n_0}\mu(E_{n_0}) = +\infty$$

$\rightarrow\leftarrow$. Logo $\mu(E_n) < +\infty$, $\forall n$

6.F) Se $f \in L^p$ e $E_n = \{x \in X \mid |f(x)| \geq n\}$, então $\mu(E_n) \to 0$, p/ $n \to \infty$.

De fato, suponha que não. Então $\exists\, \varepsilon > 0$ tal que, $\exists\, n_k \to +\infty$, $\mu(E_{n_k}) > \varepsilon$, $\forall k$
Mas daí

$$\infty > M = \int |f(x)|^p \geq \int_{E_{n_k}} |f(x)|^p \geq \int_{E_{n_k}} n_k^p = n_k^p \mu(E_{n_k})$$

$$\Rightarrow \mu(E_{n_k}) \leq M/n_k^p \quad , \forall k \quad \Rightarrow\Leftarrow \text{ com } \mu(E_{n_k}) > \varepsilon \;\; \forall k$$

Ou ainda:

$$M = \int |f(x)|^p \geq \int_{E_n} |f(x)|^p \geq \int_{E_n} n^p = n^p \mu(E_n)$$

$$\Rightarrow \mu(E_n) \leq \frac{M}{n^p}$$

$$\Rightarrow \limsup \mu(E_n) = \lim_{n\to\infty} \frac{M}{n^p} = 0$$

Mas $\liminf \mu(E_n) \geq 0$, logo segue $\square$

6.3 Seja X esp. de medida finita. Se f é mensurável seja

$$E_n = \{x \in X ; (n-1) \leq |f(x)| < n\}$$

Então

$$f \in L^p \iff \sum_{n=1}^{\infty} n^p \mu(E_n) < +\infty$$

Dem)

Note que

$$\sum_{n=1}^{\infty} n^p \chi_{E_n} \leq (|f(x)| + 1)^p$$

De fato, seja $x \in X$. Então $\exists n_0$, $x \in E_{n_0}$.
Poi

$$\sum n^p \chi_{E_n}(x) = n_0^p \chi_{E_{n_0}}(x) = n_0^p$$

Mas $x \in E_{n_0} \Rightarrow n_0 - 1 \leq |f(x)| \leq n_0$

$$\Rightarrow n_0 \leq |f(x)| + 1 \leq n_0 + 1$$

$$\Rightarrow n_0^p \leq (|f(x)| + 1)^p$$

Mas $|f(x)| \in L^p \Rightarrow |f(x)| + 1 \in L^p$, pois $\mu(X) < \infty$

$$\Rightarrow (|f(x)| + 1)^p \in L^1$$

$$\Rightarrow \sum_{n=1}^{\infty} n^p \mu(E_n) < +\infty$$

Além disso,

$$|f(x)|^p \leq \sum_{n=1}^{\infty} n^p \chi_{E_n}(x)$$

$$\Rightarrow \int |f(x)|^p \leq \sum_{n=1}^{\infty} n^p \mu(E_n) < +\infty \Rightarrow f \in L^p$$

pois $\int \sum n^p \chi_{E_n}(x) = \sum n^p \mu(E_n)$

De fato, $\underbrace{\sum_{n=1}^{N} n^p \chi_{E_n}}_{= f_N} \longrightarrow \sum_{n=1}^{\infty} n^p \chi_{E_n}$, $f_n \leq f_{n+1}$, $f_n \geq 0$

logo $\int \lim = \lim \int$ [illegible]